MODERN METHODS IN HIGH SCHOOL GEOMETRY.

"It is a remarkable fact in the history of science," says Playfair," that the oldest book of elementary geometry is still considered the best, and that the writings of Euclid, at the distance of 2000 years, continue to form the most approved introduction to the mathematical sciences." It is indeed unique, for where else in all the realm of science or philosophy can we find in use as the standard today a book uncopyrighted for twenty centuries? In the entrance examination for Oxford and Cambridge today, not only is Euclid the standard, but no demonstration is accepted that violates the Euclidian sequence. If on the continent and in America our worship has not been so exact, it is still true that no fundamental nor even extensive departure from the old Greek geometer has received any general sanction, nor has it even been attempted in the high schools until within two or three years. It is too early to predict how great the departure may become, if indeed Euclid is at any time to be displaced except in minor particulars. Indeed we may safely say that there are no present indications that the Riemannian—elliptical, positive curvature-plane space nor the Lobatchevskian—hyperbolic, negative curvature-plane space will ever displace our Euclidian—parabolic, zero curvature-plane space, even though it may appear that the plane of Euclid is not the plane of experience in the natural world, and that his 12″ axiom is not so self-evident, and perhaps not true after all.[1]

[1] Many attempts were made to demonstrate Euclid's parallel axiom. At length the Gordian knot was untied by cutting it, and in 1829 Lobatchevski, a Russian, published the proposition that *more* than one parallel to a given line can be drawn through a point without, and that the sum of two interior angles on the same side of a transversal which cuts two parallels is *more* than two right angles. Thence sprang an entirely new and yet logical geometry, in which the plane has a negative curvature. In a similar manner, later, Riemann proposed that the sum of the above mentioned angles was *less* than two right angles, thus laying foundation for a third geometry, on the basis of the positive curvature plane. It is now asserted that this latter is the plane of our experience.

And yet the prodigious mathematical activity of the last hundred years, exceeding in amount of production all previous effort from the beginning of the science, is evidence that the stagnation of geometrical teaching is to give way to activity of some sort. So eminent an authority as Professor Sylvester says: "I should rejoice to see mathematics taught with life and animation; short roads preferred to long; Euclid honorably shelved or buried 'deeper than ever plummet sounded' out of schoolboy's reach; projection, correlation, motion, introduced as aids to geometry; the mind of the student awakened by early initiation into ruling ideas of polarity, continuity, infinity, and ideas of the imaginary and inconceivable. It is this living interest in the subject which is so wanting in the traditional and mediæval modes of teaching."

Ball, the historian of mathematics, complains that Euclid's definitions and axioms are not all self-evident, that he fails to show why proofs are presented in the form they are—his demonstrations are synthetically presented, with no clue to the analysis by which they were reached—that he makes no attempt to generalize, is verbose and imperfect in classification. Another complaint is that pupils get no available knowledge of the logical processes they have used, as the logical processes are not classified nor studied, *e. g.*, how many pupils gain an idea that there are only a few methods of demonstration, some one of which is used in each proposition because that particular method best meets the needs of that proposition, and how many clearly see why one method and not another is best adapted? On the other hand, to the pupil's mind, is not the process of the proof of each proposition an isolated paragraph, offering no clue to the methods to be used in succeeding cases? Dr. Hirst, President of the British Association for the Advancement of Geometrical Teaching, said twenty years ago that we were no longer warned not to touch the edition of Euclid to which for more than a century we had paid literary homage.

On the whole, the tendency seems to be to acknowledge Euclid's "unassailable perfection," but to conclude that "in the

crowded life of today the time devoted to him can ill be spared." While the mathematical chair of one of our colleges cries: "Stick to the Euclidian method!" the head of the department in a prominent state university answers: "Rupture with the traditional Euclidian methods, align with the march of modern thought!"

In this country we have not been so blindly devoted to Euclid in his original purity, as have the schools of England. Indeed many attempts have been made to revise the order and treatment of Euclid after the manner of Legendre or Chauvenet, or in an independent attempt to reach a logical development, as was first done in the West, I think, by Professor Olney. Often these attempts have led to error, as in the conceptions of direction, distance and proportion, but out of it all we may look for improvement of a valuable kind. If we were to name the dozen most familiar texts in use today, it would be difficult to find differentiation among them sufficient to justify the existence of more than three, at most, on other grounds than that the various publishers might have a book to sell. Signs are not wanting that in variety at least we shall soon have an abundance. Between the rigid Euclidian books of the past and the extreme non-Euclidian of the present and future, we shall find a text containing a happy union of what we need of the old and modern geometry.

The term "Modern Geometry" is today used in so many senses that we shall do well to understand what it means in this paper, as applied to high school mathematics. It is more often used synonymously with synthetic geometry, but no one, I believe, urges that synthetic geometry should become the geometry of the high school though it may modify high school geometry by the introduction of many of its broader conceptions. The distinction between the old and the modern is mainly that in Euclid each proposition stands by itself, general laws are not traced; in modern geometry propositions that are merely different phases of one principle are grouped, and general conclusions reached, *e. g.*, the Pythagorean proposition is but that phase of the case

of the general triangle, in which the projection of one side on another becomes zero. Continuity, reciprocation, homology, organically connect principles which in Euclid are isolated, unrelated. Certainly if the wonderfully interesting and productive ideas of modern geometry can be speedily put in available form for high school use, the event will be hailed with delight by those who believe that "the world do move," in geometrical, as in all other knowledge. But we can hardly accept a book which soberly tells the high school boy of fifteen that "he has been led to rely more and more on his own resources of knowledge and ratiocination in the conduct of the foregoing investigations. He has now possessed himself of a large fund of concepts, and must test his ability to wield, combine and manipulate them in forging orginal proofs, always, bearing in mind the fundamental logical principle that every example of a general concept has all the marks of a general concept," and which assigns as Th. 135: "The transverse joins of two pairs of anti-homologous points of two circles meet on the power-axis of the circles." And yet the work above quoted, and others recently printed, or in print, are so full of the warmth of quickening life that we may confidently expect that a book at once progressive and usable must soon appear.

If such is the activity in the work of transformation in the text-book, we may look for a cause which will work out its effect as well in the aims and methods of *teaching* the subject. And here we are not disappointed. For methods that are various, and chaotic, and sometimes idiotic, commend us to some of the geometry teaching of the past decade. The marvelous stagnation of past centuries has been succeeded by an exhibition of well-meant acrobatics, which has been manifested as well in many other phases of recent pedagogical development, and which is always an attendant of change from the excellent past to the better future. A few original exercises, and much demonstration in the text; a little formal demonstration in the text, and much original work; no demonstration in the text; and no text at all, have been some evidences of this fermentation in the direction of the text-book.

A clear understanding of the changes that must come in the work of the high school takes us back to the elementary schools. The committee of ten urged that the child's geometrical training should begin as early as possible; in the kindergarten, if he attends one; if not, in the primary. It is claimed that geometry, and not arithmetic, is the study which possesses inherent natural qualities. It is easier for the child by experiment with models and drawings to conclude that the sum of the angles of the triangle is two right angles than to grasp the rules of fractions, *e. g.*, the rule for the multiplication and division of fractions. During all these earlier years the work is to be done without a text, and without assigned lessons, making free use of problems. At about ten or eleven years systematic instruction in concrete geometry is to begin, with constant experimental work in examples taken from the pupils environments at home, at school, and in field excursions. A very suggestive article in this line appeared in Volume XXXVIII. of the *Popular Science Monthly* and has been reprinted in some educational journals. All this work leads up to some demonstrative geometry, taken very slowly and in small doses, in the 7th and 8th grades. It is easy to see how entirely different is the aspect of high school geometry if this should be the basis of future geometric teaching. Instead of pupils to whom geometry is an unknown country, who think of an angle as two lines, and a triangle as three, we shall have to deal with those who have been on terms of intimate acquaintaince with the concepts and many of the principles of geometry.

Another important modification of methods in geometrical teaching results from the changed view of the psychology of the subject. The old view was that it was for the culture of the logical powers through the memory. The "absolutely perfect model" furnished by Euclid was to be learned. Today the tendency is to forbid the use of memory even where it is needed. The advantage of memorizing theorems, definitions and principles is very great. A habit of memorizing demonstrations is positively vicious, and must be thwarted at any cost. The

Maine Pedagogical Society has aptly stated the aim of geometry to be to cultivate, (1) attention, (2) orderly thought, (3) concise expression, (4) power of reasoning, (5) a habit of questioning. To all of these the exercise of the memory is incidental and subsidiary, though important, if exercised in the right place. The whole result must be that the pupil shall assimilate, not memorize.

The effect of this changed view of the study of geometry has been very great. Universally the exercises for demonstration by the pupils, which the British Association for the Improvement of Geometrical Teaching first effectively urged twenty years ago, have been adopted as an important part of the work. Spencer's *Inventional Geometry*, which appeared in 1876, lead the way to a practical familiarity with geometrical constructions, which that association urged should precede the theoretical study. It was to be expected that extremists would conclude that, if a little originality were a good thing, more would be better, and hence it comes to pass that recently a book has been issued containing no proofs, but only suggestions, *e. g.*, "Let ABC represent a triangle in which side AB is greater than AC.

To prove that angle C is greater than angle B:

Sug. 1. Angle C equals angle B, or is less than B, or is greater than B.

Sug. 2. If angle C equals angle B, how would AB and AC compare? Why? Does this agree with the hypotenuse? Is it possible for angle C to be equal to angle B?

Sug. 3. If the angle C is less than angle B, how would AB and AC compare? Why does this agree with the hypotenuse? Is it possible for angle C to be less than angle B?

Sug. 4. From results reached in suggestions 1, 2, 3, what must be the relation between angles C and B? etc., to the end of the chapter, and the book.

Other reformers, more thorough still, have issued texts containing not a demonstration nor definition from cover to cover. The author of one of these truly remarks that this book is not for the teacher who is lazy or lacks a thorough understanding of the subject. No one can question that, but not all will agree to his inference that others will find in it the ideal text by which to

reach the aims of geometry teaching. Those are not wanting, however, who, seeking not for themselves the glory of authorship proclaim the text, even though it be but a syllabus, an outlaw forever banished from the protection of the schoolroom, changing Plato's inscription, "Let none ignorant of geometry enter my door" to "Let none bearing a text-book enter here." At least one author prepares for all comers by making a text on the old models with numerous exercises, and issuing separately a syllabus of all the propositions.

All these experiments are to the same purpose, viz., to throw the pupil on his own resources, to cultivate his creative power and stimulate invention. It is certain that no subject of the curriculum offers such opportunities in these directions as does geometry, and a proper correlating of these aims with other legitimate ends must be the solution of the problem of methods, both in preparation of text and conduct of recitation. In my class, when I studied geometry, was a young lady of fine mind who recited the usual lesson with a fluency and inerrancy that delighted the teacher, put to shame the idle student, and aroused the envy of all. But let the then unusual lesson of original work be called, and she blushed and was silent. I cannot remember that she ever succeeded in doing an original exercise. The day has certainly come when such work is not received with favor, if indeed it is to be accepted. The conference report to the Committee of Ten but voiced the feeling of every up-to-date teacher of geometry when it said, "As soon as the student has acquired the art of vigorous demonstration, his work should cease to be merely receptive. He should begin to devise constructions and demonstrations for himself. Geometry cannot be mastered by reading the demonstrations of the text-book. By wise instruction after this method the inferior student can at least be freed from slavish dependence on his text-book, while the able student will gain power large enough to construct his own geometry." And yet with all the acknowledged merits of this innovation, I record my conviction that the virtue of added interest and power secured by this originality has been, in no small degree, offset by the

evils of departing from the rigor of the demonstrations of twenty years ago, when our books had no undemonstrated exercises. Why should this be so, and how is it to be checked?

I think *why* is clear. To successfully answer the *how* is to solve the problem of the future of geometry teaching. As to the why, we know that the laboratory of the inventor is not usually an orderly place. Strewed all about are broken pieces, and other evidences of the thousand unsuccessful experiments that were necessary before the one successful one was found. The same condition must befall the geometry class that is attempting inventive work, relying, as many teachers have insisted they must, entirely on their own inner consciousness. It required men of a good amount of ability, and not a few of them, to discover the geometrical propositions we know today, and they spent some thousands of years doing it. Our ancestors discovered many things that we need not require each of our children to evolve out of inner consciousness. We build on the past, and need not compel each pupil to follow in all details the development of the race. Some of the finest *impromptu* efforts have been previously prepared; and the inventional work of our geometry classes needs not only careful guidance but systematic, pre-original preparation, if their so-called original work is to succeed. The average youth is incapable of making his own geometry. In another direction, too, failure has been invited, in that the exercises in the book are often poorly graded, usually too difficult, and always in large part unrelated with the body of the work.

We thought we were ready to cry "Eureka!" when we hit upon the plan of giving the pupils undemonstrated exercises and inventive geometry. But we had yet to find out that pupils needed to *know how* to use them. And all our text books and many of our teachers forgot to tell the pupils how, if indeed some even knew there was a how of attacking the work except by approaching each problem as an independent puzzle. Hence this puzzle once solved was no help to the solution of the next, except for the intellectual cunning thereby acquired. Even in the study of the ordinary theorems and problems how often pupils ask, "But how

did any one ever think of such a proof?" The synthetic proofs of the text give no clue to the analysis by which the steps were involved. And then between the proofs given and the proofs to be invented the text has a great gulf fixed, with no suggestion of a bridge or ferry. To sum up in a word, there are just a few well defined methods of attack for all geometric propositions. These the class should thoroughly master and frequently, understandingly and rigidly apply, under the teacher's supervision. Enough room then remains for unlimited originality, and a degree of rationality. Not only use original exercises, not only make pupils think, but teach them how to use them, how to think.

I mention one other method—that of the use of models. In many schools an extensive use of manufactured models is constantly made; in more pasteboard models of home manufacture are provided; in some the opposite extreme is followed, and not models only but blackboard diagrams also are banished from the recitation, the class forming mental images of the figure and projecting them perhaps on the floor, or possibly better yet, from the point of view of united attention, on the teacher. My own idea is that both these extremes have in them somewhat of good, and, if too persistently used, much of evil. Certainly, if geometry is to find its way to the lower grades, the use of models is helpful and essential, though most of them may well or better be of home manufacture, worked out together with the solution of the principle. In the high school they have their place, and under present conditions no collection is likely to be too fine or too extensive.

I think I can sum up the few modern methods I have room to mention in this paper by an outline suggestion of the construction of the text I would use. I would not do without a text; I would not use a syllabus merely. Few men have shown ability to write a successful text, and the average teacher has neither the time nor the adaptability to furnish proofs by directing the class in their independent work. If this be true of the teacher what shall we say of pupils writing their own text? Pupils need constant familiarity with the most lucid, logical, elegant and rigid proofs the best text can provide. So much of the *old* should be

retained. Of the *new* have the inventive and modern. Let the book direct the pupil as he strikes out on the unknown, unproved waters of inventive work. Let the book treat of methods of attack and carefully illustrate their use. In other words let the exercises be not added leaves of the text, but an organic part of the whole, with the lines of connection clearly and constantly evident. Give us not a stream of oil and water that will not mix, nor of milk and water that clouds the whole, but a mineral spring in which the parts are in compound. Then put in enough modern geometry to widen the pupil's horizon, as Euclid would have done had these modern developments been known in his day. Let the laws of continuity, of converse, of reciprocals and homology, show the relations and interrelations of principles that must else seem isolated and confusing. Let the pupil perhaps glance into the conceptions of the elliptical and hyperbolic geometries just enough to help him realize that there are undiscovered worlds of startling interest the knowledge of the existence of which may greatly stimulate his attention to the matters presented in the Euclidian world he had supposed he lived in. By experience I know this is a force that can be used to advantage.

I might use the word *method* in the narrow sense, and give *devices* successfully used by various teachers. But orthodoxy is my doxy and heterodoxy is your doxy, and a method which you may use very successfully and pronounce very orthodox pedagogy is not unlikely to develop very heterodox pedagogy in my hands. Leave every man to his own *devices* if only his knowledge of the subject and its pedagogy and history be broad. Then, if he be a teacher, he will succeed.

Whatever the devices used I believe that through all the mistakes made, and to be made, the outcome will be modern methods in geometry that shall lead the pupil to assimilate, not to absorb, to develop, not like Topsy, "jest to grow," and that shall compel the teacher to teach not a method, not a text-book, but the *subject* in all its beauty and power and with all the marvelously good of the old and the stimulus of the broader, ever-growing new.

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